1D Basis Expansion Models for the HERA Primary Beam

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Abstract

In this memo, we study how to build a basis representation of the HERA primary beam that can be used to (e.g.) accurately model beam variations between antennae, including in the sidelobes. As a first step we consider the azimuthally-averaged primary beam (as a function of zenith angle only), using the simulated HERA primary beam from Fagnoni et al. (2019) as a realistic test case. After comparing several different choices of basis expansion, we find that the Chebyshev polynomial provides a reasonably compact and accurate representation. For example, the Chebyshev polynomial with 18 parameters can represent the azimuthally-averaged Fagnoni beam to a fractional accuracy of better than $3 \times 10^{-2}\%$ within $2\times$ FWHM, degrading to roughly 1% in the outer sidelobes. For the 2D residuals, we find that the remaining azimuthal dependence of the Fagnoni beam follows a simple oscillatory pattern that should be straightforward to model, although it is unclear whether this is representative of the primary beam of the deployed antennae.

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1. Premise of this study

We would like to find an accurate and efficient functional basis to model the 2D frequency- and polarization-dependent primary beam (PB) pattern from each HERA antenna. This will allow us to study deviations from perfect redundancy in a realistic manner, and ideally to constrain the beam patterns of the deployed antennae themselves. By 'accurate', we mean that we hope to find a basis that represents the beam with the smallest possible errors in both the mainlobe *and sidelobes*. By 'efficient', we mean that only a relatively small number of coefficients should be needed to obtain an accurate result.

As an initial exercise, we try to construct a basis that accurately models the electromagnetic simulation of the HERA primary beam from Fagnoni et al. (2019).[1] This is our current best estimate of what the primary beam pattern of a real HERA antenna should look like, and a basis that is unable to model the Fagnoni beam accurately is unlikely to be able to model the real, deployed beams accurately either. In the first instance, we study only the azimuthally-symmetric case for a single frequency channel and polarization, in the hope that this can be extended to a 2D frequency- and polarization-dependent model through extrapolation or similar.

The Fagnoni beam map is presented in healpix format with `nside=32`. The frequencies are in the range 100MHz to 200MHz, with a total of 26 frequency channels. In this study, we consider only the Stokes I HERA primary beam at a frequency of 100MHz, which is shown on a log scale in Fig. 1. We take the average along the azimuthal (ϕ) axis to represent the simulated beam as function of zenith angle (θ) only.

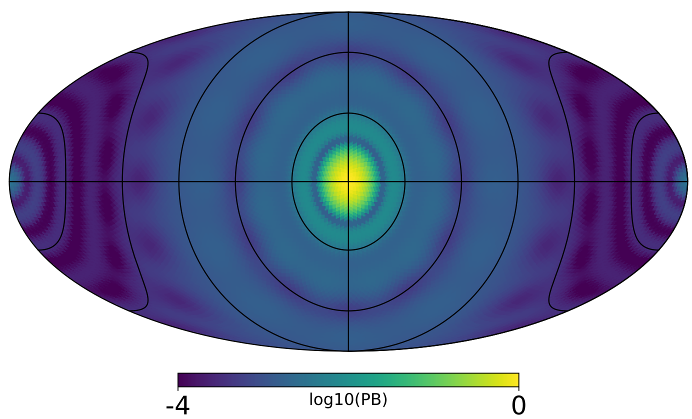

Figure 1: Healpix map of the simulated HERA primary beam ($\log_{10}$ scale) for Stokes I at 100MHz. The healpix map has `nside=32`. The black solid lines show the zenith angles 0 to 180° in intervals of 30°.

[1]Taken from the file `HERA_NF_efield.beamfits`, converted from E-field to Stokes I.

2. Comparison of polynomial fits to the primary beam

We attempted to fit the azimuthally-averaged PB with different types of polynomial, also with different numbers of parameters, to determine which representation is the most efficient. The polynomials we used for the fits were Chebyshev, Hermite, Polynomial, Legendre and Laguerre.

The Chebyshev and Legendre polynomials are orthogonal in the range $[-1, 1]$, so we transformed the zenith angle into a new variable, $x = 2\sin(\theta) - 1$ which has this range. This produced generally good results. The Hermite and the Laguerre polynomials are orthogonal in the range $[0, \infty]$ and $[-\infty, +\infty]$ respectively, prompting us to try the respective transformations $x = \tan(2\theta - \pi/2)$ and $x = \tan(\theta)$. We obtained poor results with these choices however. For the sake of uniformity, and to get the best results, we used the same variable $x = 2\sin(\theta) - 1$ for all polynomial types.

We also tried fitting the azimuthally-averaged PB with a real Fourier series,

$$\varepsilon(\theta) = \sum_{n=0}^{N-1} a_n \cos(2\pi n\theta/P) + b_n \sin(2\pi n\theta/P), \tag{1}$$

where $P = \pi$ is the period, and $2 \times N$ is the total number of parameters for the Fourier basis.[2]

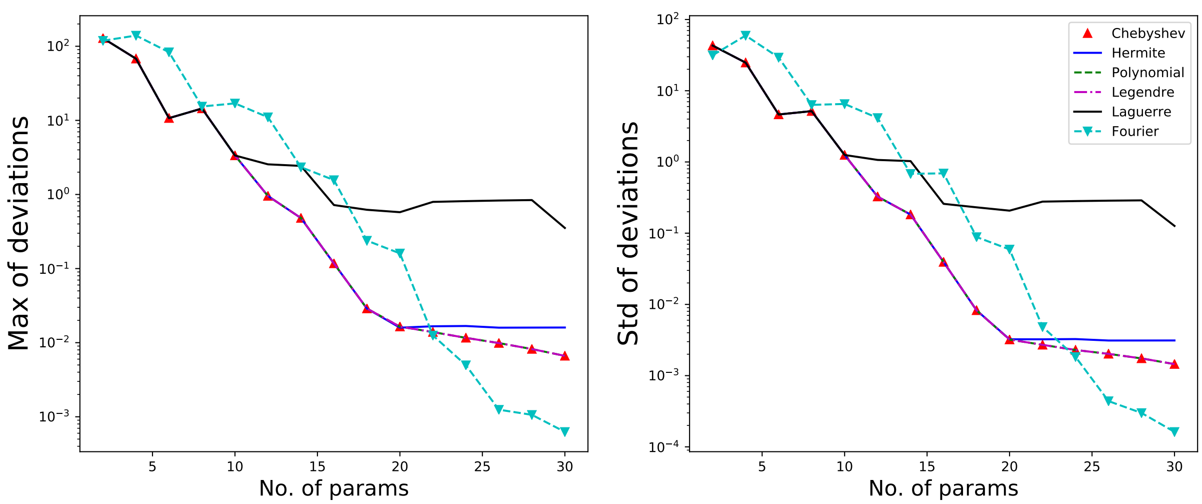

Figure 2: *(Left):* Maximum of the absolute fractional deviation between the primary beam and fitting functions of different number of parameters. *(Right):* Standard deviation of the fractional deviation.

Fig. 2 summarizes the results for all types of fitting function as a function of total number of parameters. The left panel shows the maximum absolute fractional deviation (the 'worst case'), and the right panel shows the standard deviation of the fractional deviation (the 'typical' value). For the most part, the values of the maximum and standard deviation decrease with increasing the number of parameters, with most functions converging after a while. The results for Chebyshev, Polynomial, and Legendre are quite similar, and the max./std. dev. rapidly converge to better than 3×10^{-2} at orders ≥ 18, with a slower improvement with increasing the number of parameters beyond this. The Hermite polynomials also have a similar behaviour, but plateau after 20 number of parameters, instead of continuing to improve. The Laguerre polynomials produce generally poor results regardless of the number of parameters.

[2]We did not include the $\theta = \pi/2$ point in the Fourier fits, as this resulted in oscillatory behavior that degraded the fits (possibly due to ringing or similar; more investigation is needed). To make the θ range similar for all cases, we remove $\theta = \pi/2$ in this study.

Also, the Fourier series is not good enough as compared with the Chebyshev/Polynomial/Legendre at smaller number of parameters, but gives best results when the number of parameters is more than 22.

Based on these results, we use the Chebyshev polynomial with 18 number of parameters for the rest of the analysis, as this gives very reasonable results for our purposes. If we increase the number of parameters the result can improve further, but at a much slower rate. Example results for the Chebyshev and Fourier fits at particular number of parameters are shown in Fig. 3 and Fig. 4 respectively.

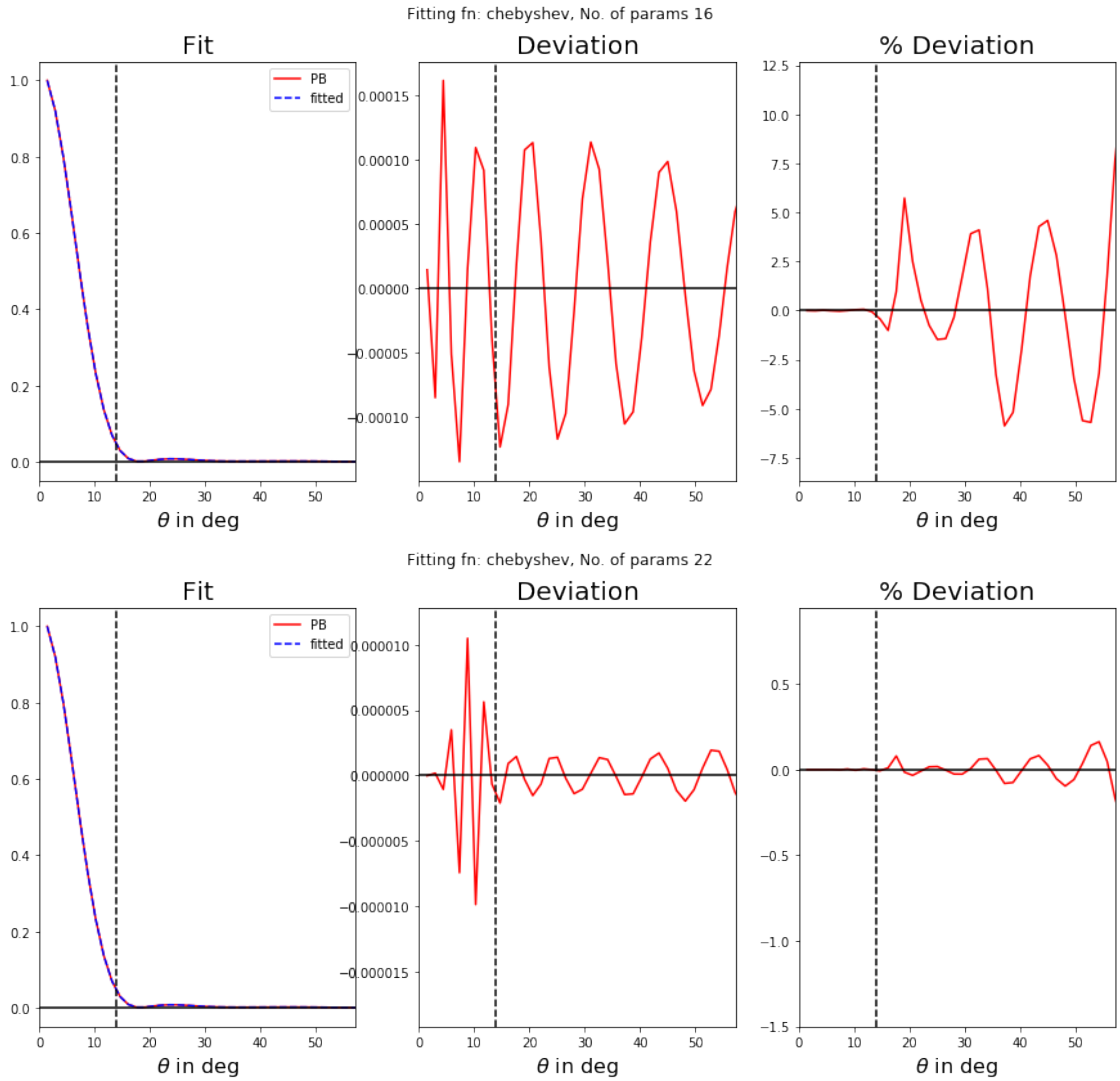

Figure 3: Results for Chebyshev polynomial fitting of the residual primary beam. The upper and lower panels are for 16 and 22 number of parameters respectively. *(Left):* Red solid and blue dashed lines show the azimuthally-averaged PB and the best-fit polynomial respectively. *(Middle):* Deviation of the best fit polynomial with respect to the PB. *(Right):* Percentage deviation of the polynomial fit with respect to the PB. The dashed black vertical line denotes 2× FWHM.

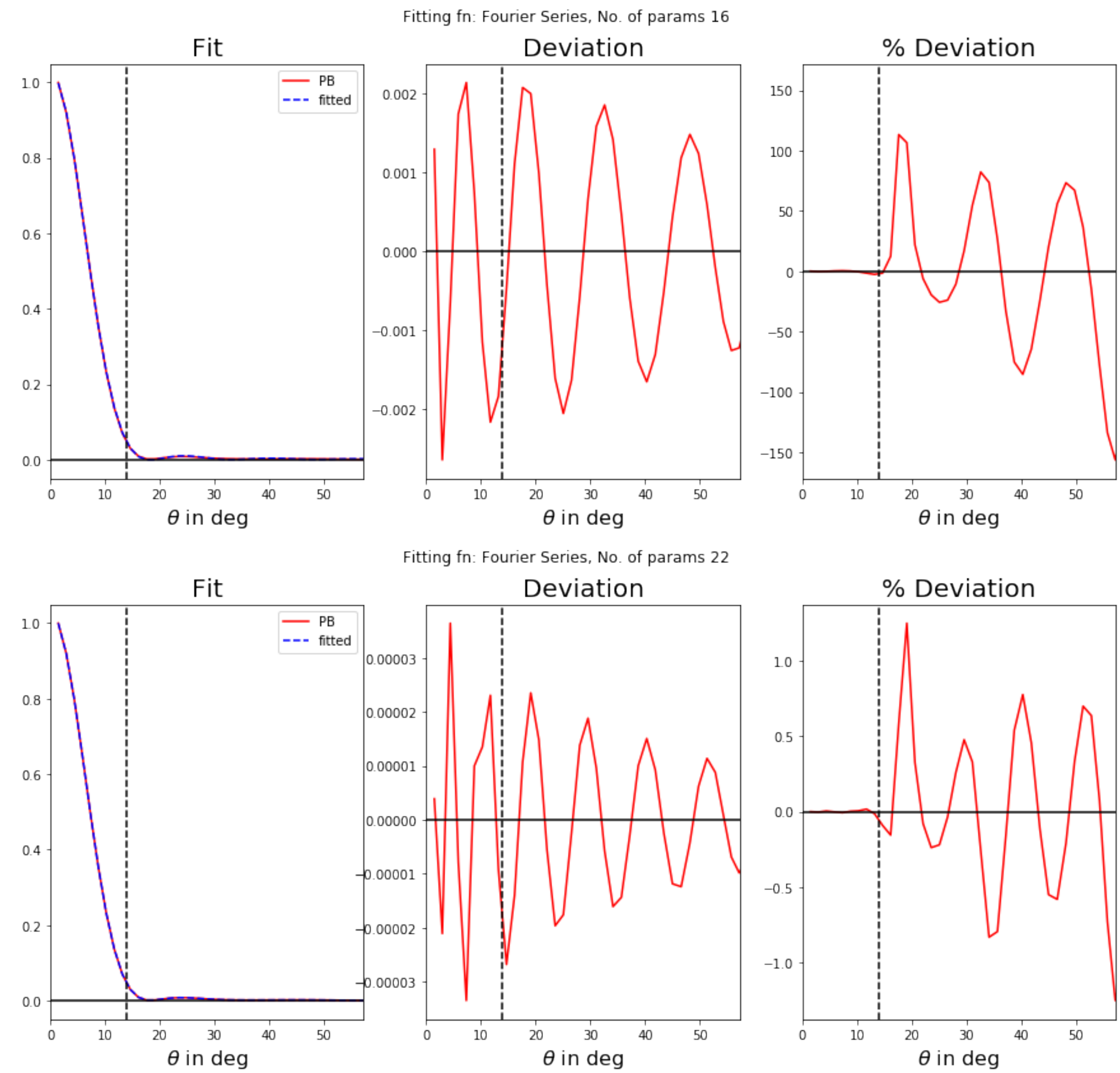

Figure 4: Same as Fig. 3 but for the Fourier series. Here we also see similar behaviour to the Chebyshev polynomial, but the percentage errors are more as compared with Chebyshev polynomial.

3. Chebyshev fit to the primary beam

We now present the Chebyshev polynomial fit to the simulated HERA PB. The Chebyshev polynomial represents the main lobe and the outer regions of the beam quite well in this case.

For our baseline model, we use a total 18 parameters for the Chebyshev polynomial (Fig. 5). The upper left panel shows the azimuthally-averaged simulated PB (red solid line) and the best-fit model (blue dashed line). The second upper left panel shows the same but on a log scale. We see that the fitted beam matches the simulated beam very accurately, even in the sidelobe region.

The second right panel of Fig. 5 shows the difference between the simulated beam and the best-fit model, and the right panel shows the percentage deviation. We see that the fractional deviation is $< 3 \times 10^{-2}\%$ within $2\times$ FWHM (black dashed line), and less than $< 1\%$ out to $\theta < 60°$. The lower panels show the same quantities but without azimuthal averaging, thus quantifying the asymmetry along the azimuth axis. Here, we see that the asymmetry is not very important within $2\times$ FWHM, where the fractional deviation remains small as in the azimuthally-averaged case. The asymmetry becomes important at wider zenith angles however, with the deviation growing to almost 20% at 60°.

We also show the 2D residual of the best-fit primary beam model with respect to the Healpix map of the Fagnoni beam that was shown in Fig. 1. The left panel of Fig. 6 shows a 2D map of the best-fit Chebyshev primary beam, while the right panel shows the percentage deviation of the fitted beam with respect to the simulated one. As shown in lower right panel of Fig. 5, the fractional deviation is very small up to 15° (around twice the FWHM), but then increases significantly, to around 20% at around 60°. There is clearly a lot of regular structure in the residual however, notably an oscillatory pattern along ϕ direction.

To illustrate the regularity of this pattern, Fig. 8 shows the fractional deviation as a function of azimuth angle ϕ for a few different values of the zenith angle, θ. Broadly speaking, the amplitude of the oscillation increases with zenith angle, but seems to be a relatively simple function of ϕ (presumably involving only a small number of Fourier modes).

We also make a simple attempt at introducing a frequency dependence into the beam model, by scaling the zenith angle by a power law in frequency:

$$x(\nu) = 2\sin(\theta/f_\nu) - 1; \qquad f_\nu = \left(\frac{\nu}{\nu_{\rm ref}}\right)^\beta . \tag{2}$$

The power law spectral index is chosen by fitting to the width of the mainlobe of the Fagnoni beam as a function of frequency, yielding $\beta = -0.6975$ for $\nu_{\rm ref} = 100$ MHz when we fit to only the 100 and 200 MHz channels. A comparison of the frequency-dependent model, with Chebyshev coefficients fitted only at 100 MHz, to the actual frequency dependence of the Fagnoni beam is shown in Fig. 8. The frequency dependence of the sidelobes is clearly more complicated than what we can model with a simple stretch factor.

Finally, the Chebyshev coefficients for an 18-parameter fit to the Fagnoni beam at 100 MHz are:

```
coeffs = np.array([
        2.35088101e-01, -4.20162599e-01,  2.99189140e-01, -1.54189057e-01,
        3.38651457e-02,  3.46936067e-02, -4.98838130e-02,  3.23054464e-02,
       -7.56006552e-03, -7.24620596e-03,  7.99563166e-03, -2.78125602e-03,
       -8.19945835e-04,  1.13791191e-03, -1.24301372e-04, -3.74808752e-04,
        1.93997376e-04, -1.72012040e-05
        ])
```

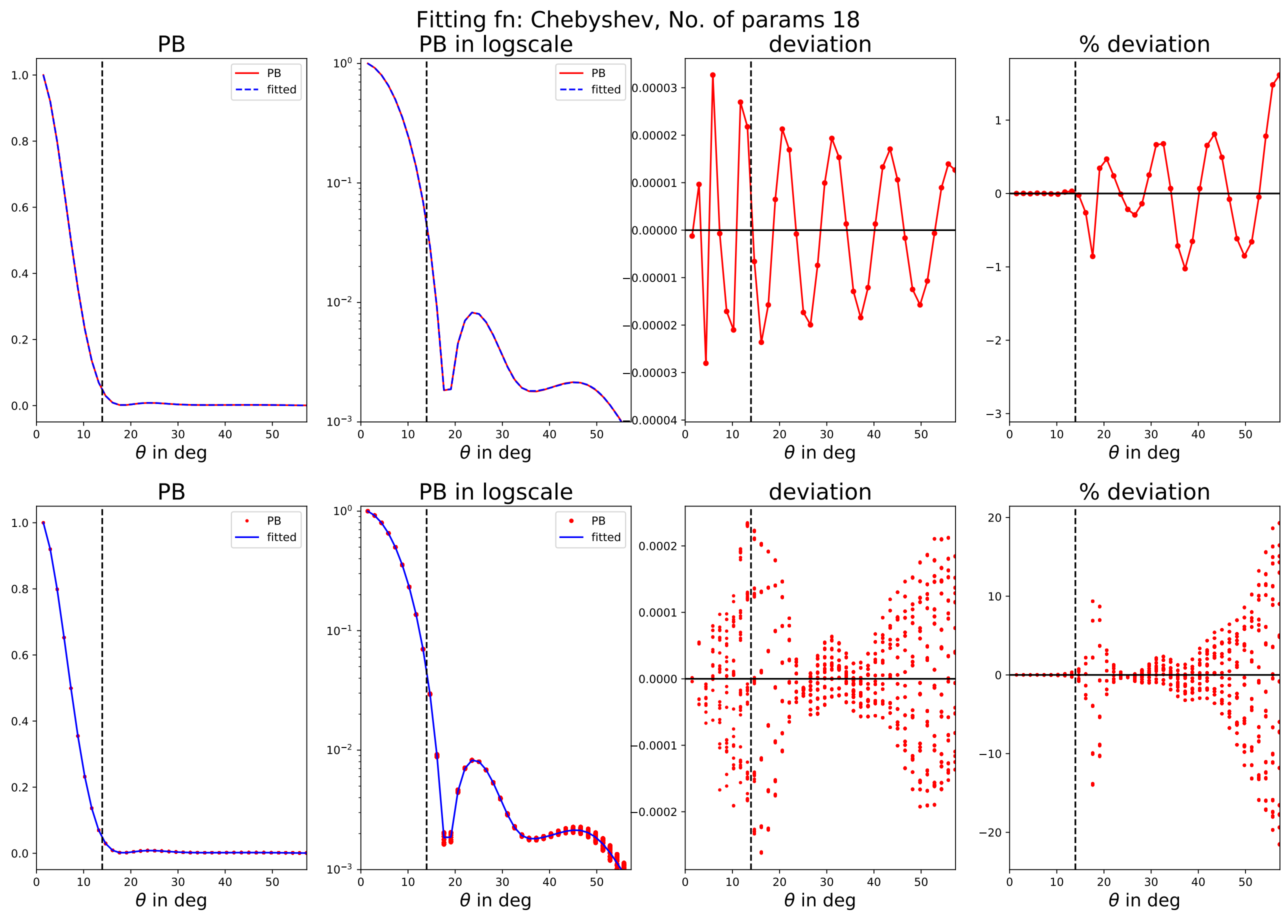

Figure 5: *(Upper left):* Plot showing the azimuthally-averaged Fagnoni primary beam (red solid line) and best-fit Chebyshev polynomial with 18 parameters (blue dashed line) as a function of θ. *(Upper 2nd from left):* Same as the previous panel, but with a log scale. We see that the Chebyshev polynomial fits quite well in this case. *(Upper 2nd from right):* Difference between the simulated primary beam and best-fit Chebyshev polynomial. *(Upper right):* Fractional (percentage) difference of the best-fit Chebyshev polynomial from the simulated primary beam. The fractional deviation is significantly smaller, $< 3 \times 10^{-2}\%$ within $2\times$ FWHM, and less than $< 1\%$ out to $\theta < 60^{\circ}$. *(Lower panels:)* Same as above, but without azimuthal averaging. Here, The fractional deviation remains small within $2\times$ FWHM, but the deviation growing to almost 20% at 60°.

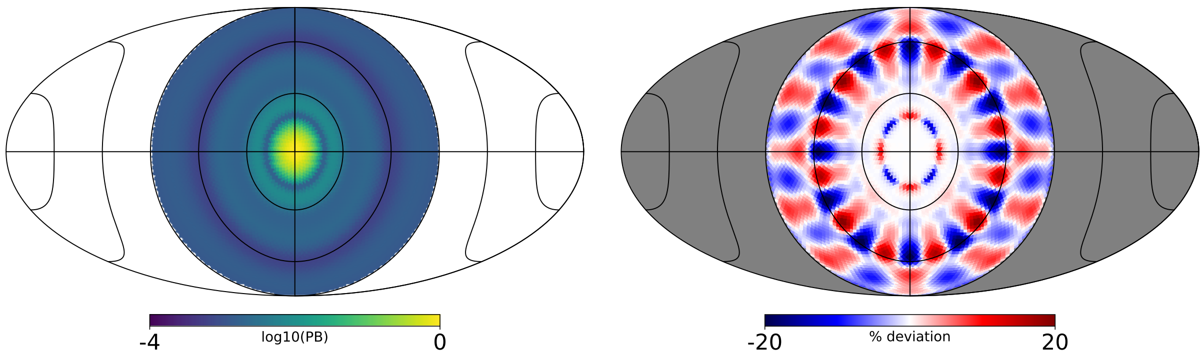

Figure 6: Healpix representation of the fitted primary beam. The left panel is the best-fit Chebyshev model. The right panel is the fractional deviation of the fitted beam with respect to the simulated Fagnoni primary beam. The red and blue colors represent the positive and negative values of the percentage deviation with respect to the simulated beam. The black solid lines in each panel show the zenith angle at 30° intervals.

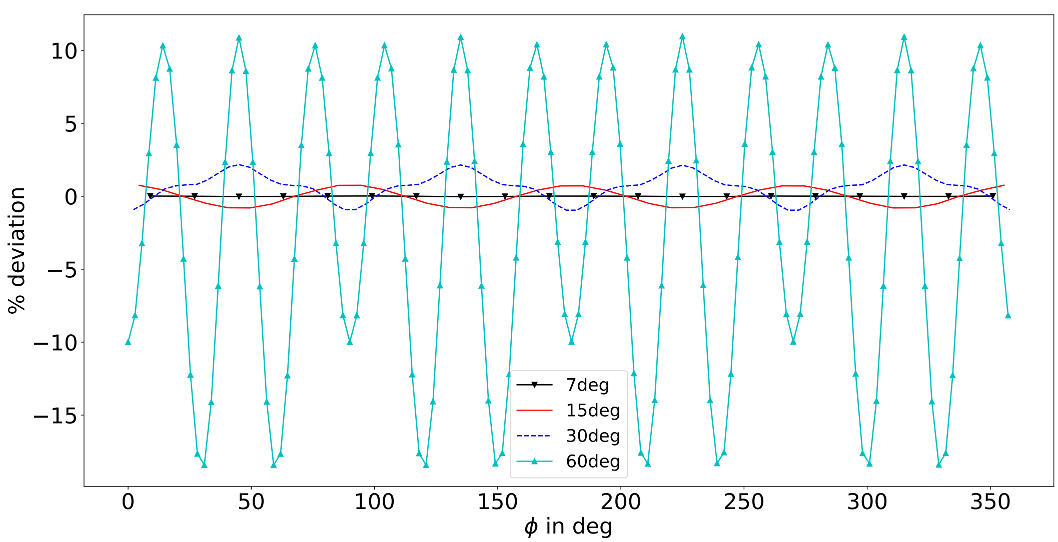

Figure 7: Percentage deviation of the 1D (Chebyshev polynomial) model from the Fagnoni beam as a function of azimuth angle ϕ, for a selection of zenith angles.

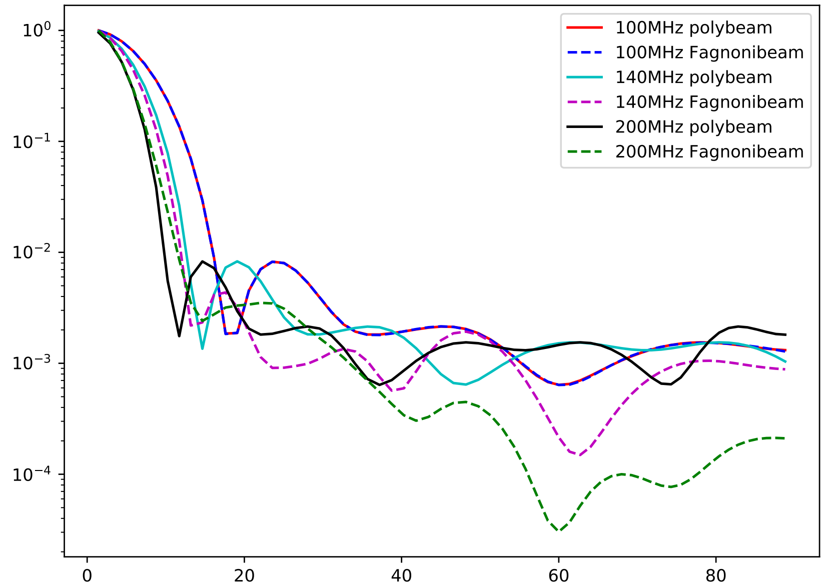

Figure 8: Azimuthally-averaged primary beam at 100, 140, 200 MHz. Solid lines use a simple power-law stretch of the beam with frequency (for the best-fit Chebyshev polynomial at 100 MHz); dashed lines show the Fagnoni beam at each frequency.

4. Summary and conclusions

An accurate model for the HERA primary beam is required for absolute gain calibration and normalization of the power spectrum. Here, we present a one-dimensional model of the HERA primary beam as a function of zenith angle only. Although there is some asymmetry along the azimuth axis in the outer regions of the beam, we considered only a 1D model for the time being.

We fit the azimuthally-averaged PB with a selection of different types of polynomial and a Fourier series. The results for the Chebyshev, Legendre, and Polynomial bases are quite similar, while the results for the Hermite, Laguerre, and Fourier series produce markedly worse fits or are slower to converge.

Consequently, we adopted the Chebyshev polynomial with 18 parameters to fit the azimuthally-averaged Fagnoni primary beam simulation. After performing the fit, a fractional deviation of less than $3 \times 10^{-2}\%$ was achieved within twice the FWHM, and less than 1% was achieved in the sidelobes. While the percentage deviation increases when we consider the non-averaged (azimuth-dependent) primary beam at wide angles, it remains quite good (around 20%) at these higher zenith angles. We also see that there is a regular oscillatory pattern in the 2D residual, which we expect can be accounted for with a simple extension of the 1D model. It is unclear whether the Fagnoni beam model is representative of the true level of asymmetry of the deployed HERA antenna beams however, and so a more sophisticated azimuth-dependent model may be required in practise.

References

N. Fagnoni et al. Electrical and electromagnetic co-simulations of the HERA Phase I receiver system including the effects of mutual coupling. *arXiv*, art. arXiv:1908.02383, Aug 2019.