A Comparison of Two Absolute Calibration Methods

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1 Introduction

This memo is intended as a supplement to Byrne et al. 2019 (in review), available at `https://arxiv.org/abs/1811.01378`. The work presented in this memo follows the conventions developed in that paper.

2 Absolute Calibration from Sky-Based Calibration

One method of absolute calibration uses sky-based calibration solutions. Full sky-based calibration is implemented by varying g_i^{sky} to minimize

$$\chi^2_{\text{sky}}(f) = \sum_{jk} \frac{|v_{jk}(f) - g_j^{\text{sky}}(f) g_k^{\text{sky}*}(f) m_{jk}(f)|^2}{\sigma^2_{jk}(f)}. \tag{1}$$

The absolute calibration parameters A, Δ_x, and Δ_y are then fit from the maximum-likelihood sky-based gains $\hat{g_i}^{\text{sky}}$:

$$\hat{A}(f) = \frac{1}{N} \sum_{j=0}^{N} |\hat{g_j}^{\text{sky}}(f)| \tag{2}$$

and Δ, Δ_x, and Δ_y are varied to minimize

$$\chi^2_\phi(f) = \sum_{j=0}^{N} \Big(\text{Arg}[\hat{g_j}^{\text{sky}}(f)] - \Delta(f) - \Delta_x(f) x_j - \Delta_y(f) y_j \Big)^2. \tag{3}$$

Here (x_j, y_j) is the position of antenna j.

3 Optimal Absolute Calibration

An optimal absolute calibration method bypasses sky-based calibration to calculate the absolute calibration parameters directly from the χ-squared. Here the χ-squared is

$$\chi^2 = \sum_\alpha \sum_{\{k,m\}_\alpha} \frac{1}{\sigma^2_{km}} \left| v_{km} - A^2 e^{-i(\Delta_x x_\alpha + \Delta_y y_\alpha)} \hat{h_k} \hat{h_m^*} m_\alpha \right|^2 \tag{4}$$

where α denotes a redundant baseline set and $\{k,m\}_\alpha$ are the antenna pairs that contribute to that set. At this point we assume relative calibration has already been performed, and $\{\hat{h}_k\}$ are the maximum-likelihood relative calibration gains. We also assume that the overall phase Δ has been set, e.g. from a reference antenna.

We solve for each of the absolute calibration parameters by analytically minimizing the χ-squared. Taking the derivative with respect to A gives:

$$
\begin{aligned}
\frac{\partial \chi^2}{\partial A} =& \sum_\alpha \sum_{\{k,m\}_\alpha} \frac{1}{\sigma_{km}^2} \left(\frac{\partial}{\partial A} [\mathrm{Re}(v_{km}) - A^2 \,\mathrm{Re}(e^{-i(\Delta_x x_\alpha + \Delta_y y_\alpha)} \hat{h}_k \hat{h}_m^* m_\alpha)]^2 \right. \\
& \left. + \frac{\partial}{\partial A} [\mathrm{Im}(v_{km}) - A^2 \,\mathrm{Im}(e^{-i(\Delta_x x_\alpha + \Delta_y y_\alpha)} \hat{h}_k \hat{h}_m^* m_\alpha)]^2 \right) \\
=& \sum_\alpha \sum_{\{k,m\}_\alpha} \frac{-4}{\sigma_{km}^2} A \left[-A^2 e^{-i(\Delta_x x_\alpha + \Delta_y y_\alpha)} \hat{h}_k \hat{h}_m^* m_\alpha \right. \\
& + \mathrm{Re}(v_{km}) \,\mathrm{Re}(e^{-i(\Delta_x x_\alpha + \Delta_y y_\alpha)} \hat{h}_k \hat{h}_m^* m_\alpha) \\
& \left. + \mathrm{Im}(v_{km}) \,\mathrm{Im}(e^{-i(\Delta_x x_\alpha + \Delta_y y_\alpha)} \hat{h}_k \hat{h}_m^* m_\alpha) \right] \\
=& \sum_\alpha \sum_{\{k,m\}_\alpha} \frac{-4}{\sigma_{km}^2} A \left[-A^2 e^{-i(\Delta_x x_\alpha + \Delta_y y_\alpha)} \hat{h}_k \hat{h}_m^* m_\alpha \right. \\
& \left. + \,\mathrm{Re}(v_{km}^* e^{-i(\Delta_x x_\alpha + \Delta_y y_\alpha)} \hat{h}_k \hat{h}_m^* m_\alpha) \right].
\end{aligned}
\tag{5}
$$

Now setting $\frac{\partial X^2}{\partial A} = 0$ gives

$$
A^2 = \frac{\sum_\alpha \sum_{\{k,m\}_\alpha} \frac{1}{\sigma_{km}^2} \mathrm{Re}(v_{km}^* e^{-i(\Delta_x x_\alpha + \Delta_y y_\alpha)} \hat{h}_k \hat{h}_m^* m_\alpha)}{\sum_\alpha \sum_{\{k,m\}_\alpha} \frac{1}{\sigma_{km}^2} e^{-i(\Delta_x x_\alpha + \Delta_y y_\alpha)} \hat{h}_k \hat{h}_m^* m_\alpha}. \tag{6}
$$

Taking the derivative of χ-squared with respect to Δ_x gives

$$
\frac{\partial \chi^2}{\partial \Delta_x} = \sum_\alpha \sum_{\{k,m\}_\alpha} \frac{-2}{\sigma_{km}^2} A^2 x_\alpha \,\mathrm{Im}(v_{km}^* e^{-i(\Delta_x x_\alpha + \Delta_y y_\alpha)} \hat{h}_k \hat{h}_m^* m_\alpha) \tag{7}
$$

and setting $\frac{\partial X^2}{\partial \Delta_x} = 0$ gives

$$
\begin{aligned}
& \sum_\alpha x_\alpha \cos(\Delta_x x_\alpha + \Delta_y y_\alpha) \sum_{\{k,m\}_\alpha} \frac{1}{\sigma_{km}^2} \mathrm{Im}(v_{km}^* \hat{h}_k \hat{h}_m^* m_\alpha) \\
&= \sum_\alpha x_\alpha \sin(\Delta_x x_\alpha + \Delta_y y_\alpha) \sum_{\{k,m\}_\alpha} \frac{1}{\sigma_{km}^2} \mathrm{Re}(v_{km}^* \hat{h}_k \hat{h}_m^* m_\alpha).
\end{aligned}
\tag{8}
$$

We assume that Δ_x and Δ_y are small and exploit that assumption to Taylor expand around $\Delta_x = 0$ and $\Delta_y = 0$:

$$
\begin{aligned}
& \sum_\alpha \sum_{\{k,m\}_\alpha} \frac{1}{\sigma_{km}^2} x_\alpha \,\mathrm{Im}(v_{km}^* \hat{h}_k \hat{h}_m^* m_\alpha) \\
&= \Delta_x \sum_\alpha \sum_{\{k,m\}_\alpha} \frac{1}{\sigma_{km}^2} x_\alpha^2 \,\mathrm{Re}(v_{km}^* \hat{h}_k \hat{h}_m^* m_\alpha) \\
&+ \Delta_y \sum_\alpha \sum_{\{k,m\}_\alpha} \frac{1}{\sigma_{km}^2} x_\alpha y_\alpha \,\mathrm{Re}(v_{km}^* \hat{h}_k \hat{h}_m^* m_\alpha).
\end{aligned}
\tag{9}
$$

Likewise, taking the derivative of χ-squared with respect to Δ_y gives

$$\frac{\partial \chi^2}{\partial \Delta_y} = \sum_\alpha \sum_{\{k,m\}_\alpha} \frac{-2}{\sigma_{km}^2} A^2 y_\alpha \, \mathrm{Im}(v_{km}^* e^{-i(\Delta_x x_\alpha + \Delta_y y_\alpha)} \hat{h_k} \hat{h_m^*} m_\alpha) \tag{10}$$

and setting $\frac{\partial \chi^2}{\partial \Delta_y} = 0$ gives

$$\begin{aligned} &\sum_\alpha y_\alpha \cos(\Delta_x x_\alpha + \Delta_y y_\alpha) \sum_{\{k,m\}_\alpha} \frac{1}{\sigma_{km}^2} \mathrm{Im}(v_{km}^* \hat{h_k} \hat{h_m^*} m_\alpha) \\ &= \sum_\alpha y_\alpha \sin(\Delta_x x_\alpha + \Delta_y y_\alpha) \sum_{\{k,m\}_\alpha} \frac{1}{\sigma_{km}^2} \mathrm{Re}(v_{km}^* \hat{h_k} \hat{h_m^*} m_\alpha). \end{aligned} \tag{11}$$

Once again Taylor expanding around $\Delta_x = 0$ and $\Delta_y = 0$ gives

$$\begin{aligned} &\sum_\alpha \sum_{\{k,m\}_\alpha} \frac{1}{\sigma_{km}^2} y_\alpha \, \mathrm{Im}(v_{km}^* \hat{h_k} \hat{h_m^*} m_\alpha) \\ &= \Delta_x \sum_\alpha \sum_{\{k,m\}_\alpha} \frac{1}{\sigma_{km}^2} x_\alpha y_\alpha \, \mathrm{Re}(v_{km}^* \hat{h_k} \hat{h_m^*} m_\alpha) \\ &+ \Delta_y \sum_\alpha \sum_{\{k,m\}_\alpha} \frac{1}{\sigma_{km}^2} y_\alpha^2 \, \mathrm{Re}(v_{km}^* \hat{h_k} \hat{h_m^*} m_\alpha). \end{aligned} \tag{12}$$

Solving Equations 9 and 12 together gives the maximum-likelihood phase gradient parameters $\hat{\Delta_x}(f)$ and $\hat{\Delta_y}(f)$. Plugging those values into Equation 6 then gives the maximum-likelihood amplitude parameters $\hat{A}(f)$.

4 Comparing Absolute Calibration Methods

The two absolute calibration methods described here are not mathematically equivalent, but simulations indicate that they are effectively equivalent. To show this, we simulate a hexagonal array of 331 antennas (see Byrne et al. 2019, preprint available at `https://arxiv.org/abs/1811.01378`). We use the FHD analysis pipeline to simulate data visibilities from 51,821 sources from the GLEAM EoR-0 field and simulate model visibilities from the 4,000 brightest of those sources in apparent brightness. We then perform absolute calibration using both methods described in this memo. As the true gains in this simulation are 1, we represent perfect redundancy by letting $\hat{h}_i = 1$ for all antennas i.

Figure 1 plots the parameter $\hat{A}(f)$. The solid blue line was calculated from sky-based calibration solutions, and the dashed blue line was calculated using the optimal absolute calibration method.

Likewise, Figures 2 and 3 plot the parameters $\hat{\Delta_x}$ and $\hat{\Delta_y}$, respectively. Once again, the solid blue lines indicate the parameters calculated from sky-based calibration solutions and the dashed blue lines were calculated with the optimal absolute calibration method.

These plots show that the two absolute calibration methods are effectively equivalent, and that discrepancies between the two methods are at least two orders-of-magnitude less than the errors in the absolute calibration parameters from sky model

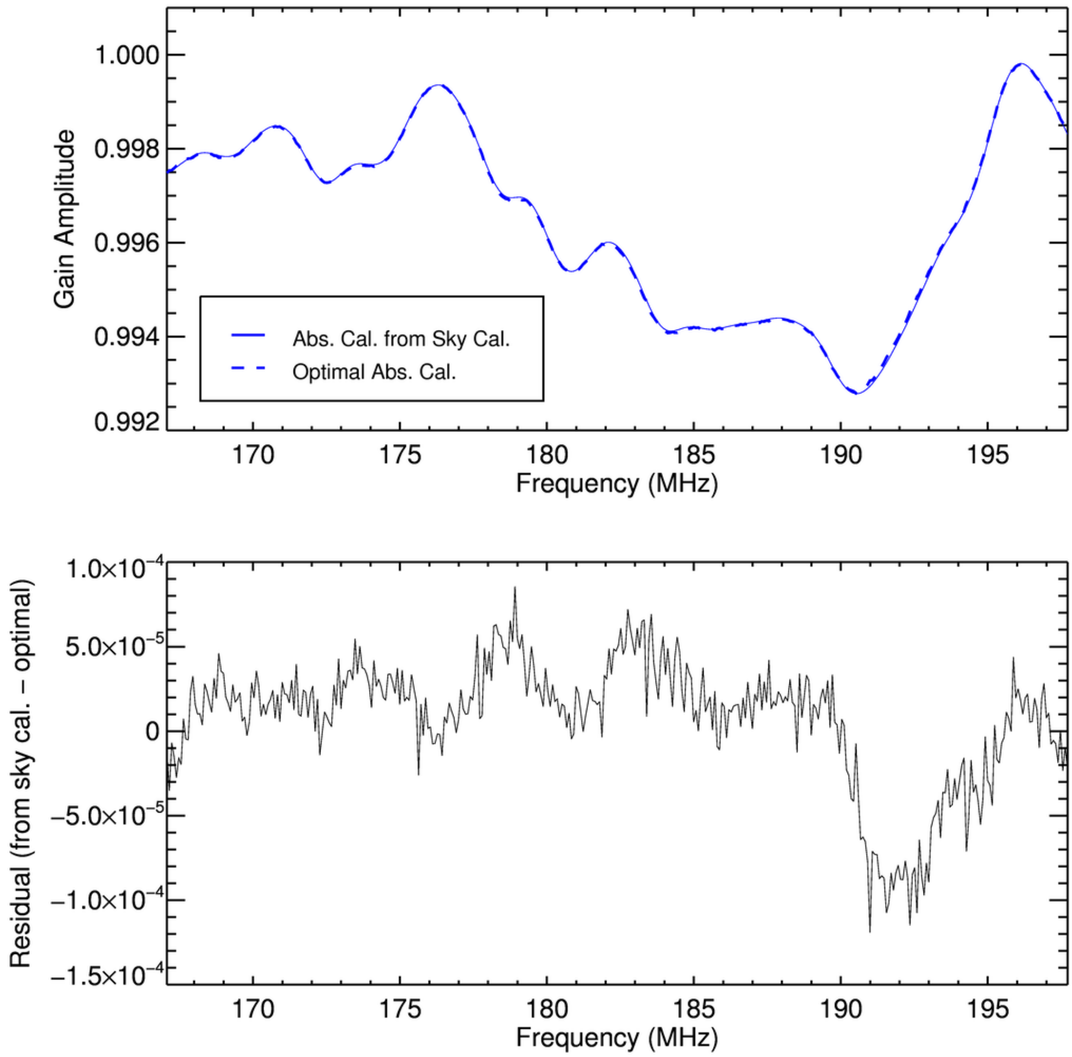
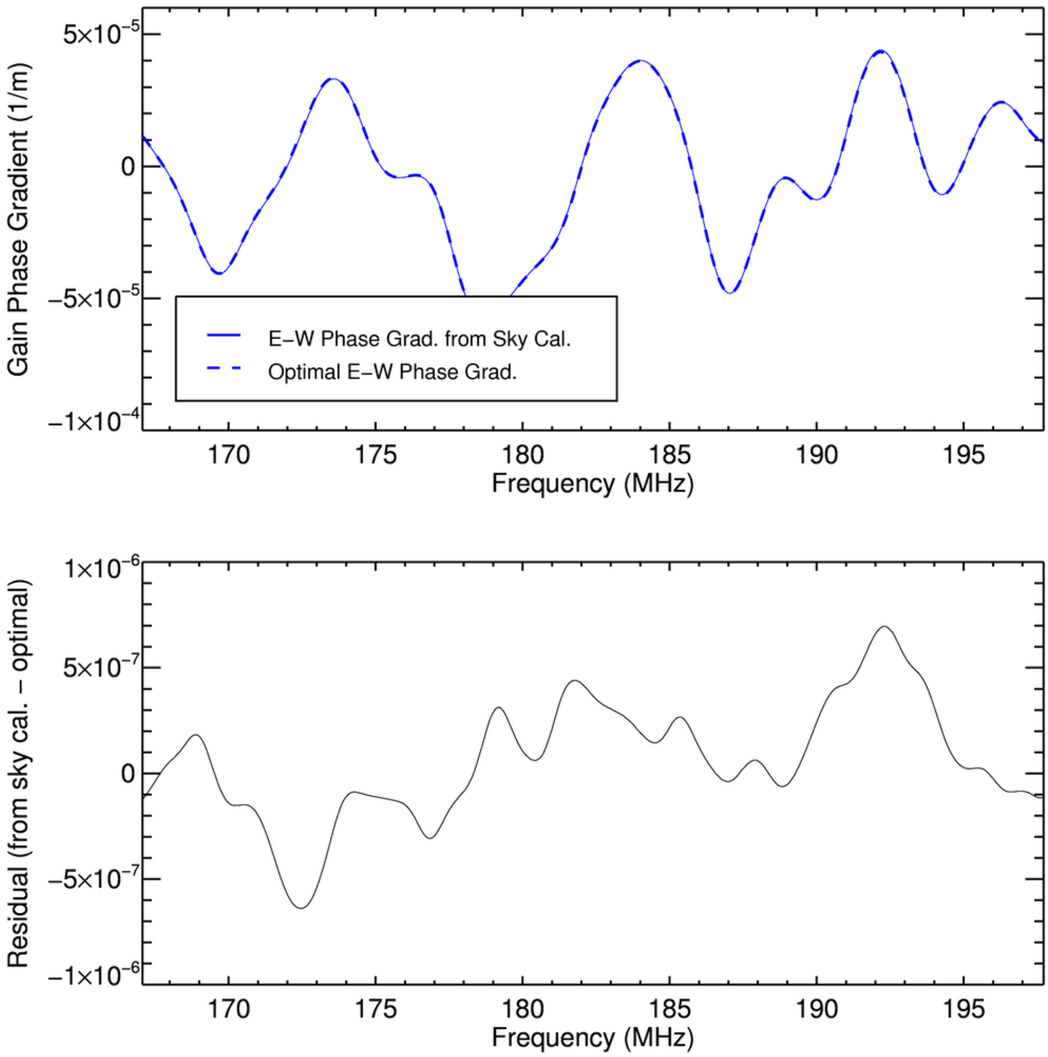
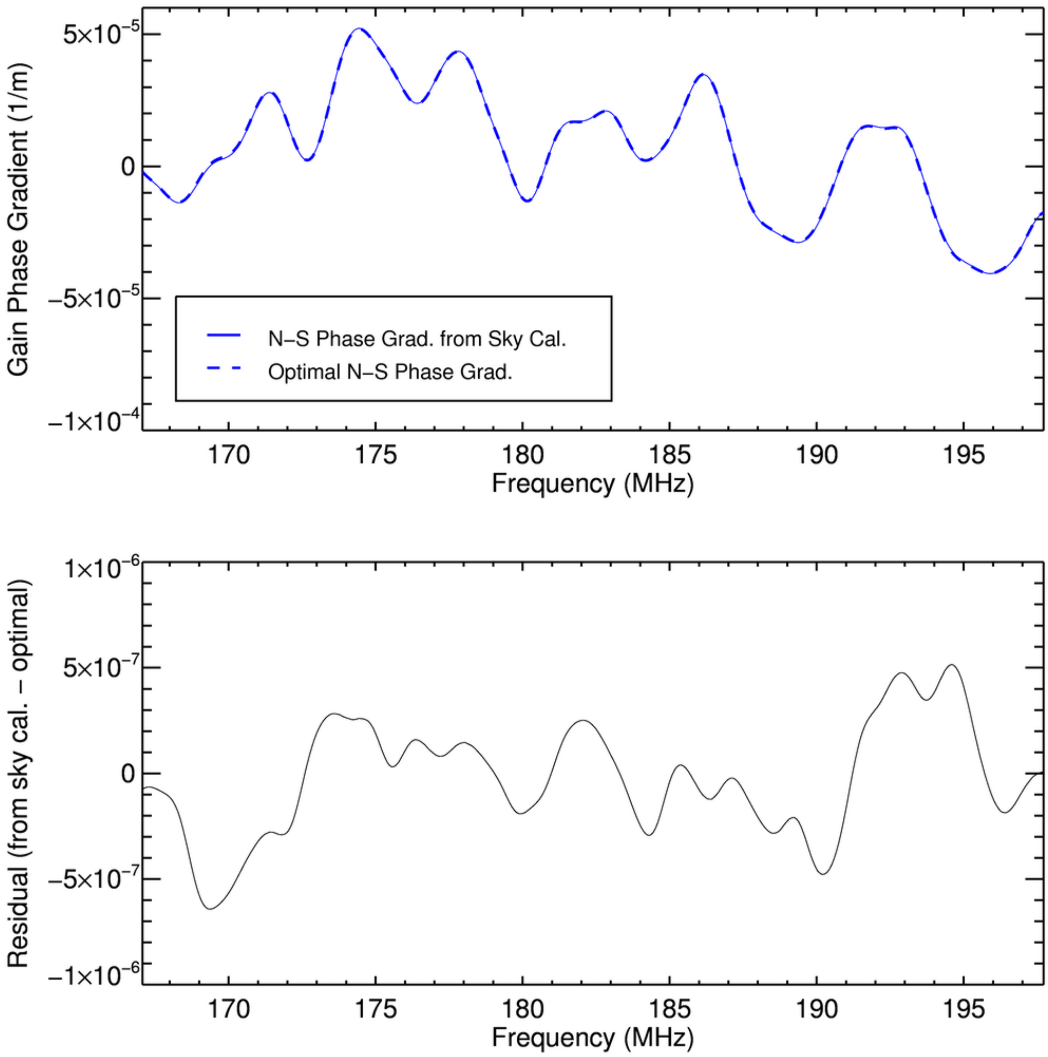

1.000
0.998
0.996
0.994
0.992
Gain Amplitude
Abs. Cal. from Sky Cal.
Optimal Abs. Cal.
170
175
180
185
190
195
Frequency (MHz)
1.0×10^−4
5.0×10^−5
0
−5.0×10^−5
−1.0×10^−4
−1.5×10^−4
Residual (from sky cal. − optimal)
170
175
180
185
190
195
Frequency (MHz)

Figure 1



5×10⁻⁵
0
−5×10⁻⁵
−1×10⁻⁴
Gain Phase Gradient (1/m)
E−W Phase Grad. from Sky Cal.
Optimal E−W Phase Grad.
170
175
180
185
190
195
Frequency (MHz)
1×10⁻⁶
5×10⁻⁷
0
−5×10⁻⁷
−1×10⁻⁶
Residual (from sky cal. − optimal)
170
175
180
185
190
195
Frequency (MHz)

Figure 2



5×10^{-5}
0
−5×10^{-5}
−1×10^{-4}
Gain Phase Gradient (1/m)
N−S Phase Grad. from Sky Cal.
Optimal N−S Phase Grad.
170
175
180
185
190
195
Frequency (MHz)
1×10^{-6}
5×10^{-7}
0
−5×10^{-7}
−1×10^{-6}
Residual (from sky cal. − optimal)
170
175
180
185
190
195
Frequency (MHz)

Figure 3

incompleteness.