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Normalisation of Power Spectrum and Noise Power

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1 INTRODUCTION

This memo presents a comprehensive overview and derivation of the power spectrum normalisation, along with a form for the thermal noise power. Its purpose is to bring together the various derivations to date, eg. Parsons et al. (2012), Parsons et al. (2014, hereafter P14a), Parsons (2017, hereafter P17) and Kolopanis et al. (2019), and clarify their various points of difference. To this end, we include a table of assumptions, Table 1, to aide in understanding which assumptions are made at which point of the derivation, and to serve as a reference for branching off derivations which break these assumptions.

For clarity of notation, we will use over-tilde throughout to represent Fourier-space functions (i.e. in uv- or k-space). Angle-brackets $\langle\rangle$ will represent *ensemble* averages (i.e. the expectation over an infinite number of realisations of the indicated process). The Var and Cov operators will similarly refer to theoretical ensemble quantities, where applicable. Vector-valued quantities will be formatted as upright-bold characters, eg. $\mathbf{r}$. Matrices will be formatted as upper-case vectors, eg. $\mathbf{C}$.

2 POWER SPECTRUM NORMALISATION

2.1 Delay Transform

Under the flat-sky and delay approximations, a measured Fourier-space visibility (of a single baseline) may be written

$$\tilde{V}(u,v,\eta) = \int dl\,dm\,d\nu A(l,m,\nu)\phi(\nu)I(l,m,\nu)e^{-2\pi i(ul+vm+\eta\nu)}, \tag{1}$$

where (l,m) are the sin-angle co-ordinates of the sky phased to a given centre, A is the beam-response pattern of the antennas in the baseline, ϕ is an arbitrary taper function applied by the analyst, and I is the specific intensity of radiation in a given direction.

The specific intensity of the sky is typically measured in Jy per steradian, such that the Fourier visibility $\tilde{V}$ has units of JyHz. The cosmological power spectrum is typically expressed in terms of brightness temperature (in units of mK), which is related to specific intensity by

$$\begin{aligned} T_b &= 10^{-26}\frac{c^2}{2k_B\nu^2}\cdot I_\nu \cdot \frac{10^3\text{mK}}{\text{K}}\cdot\frac{\text{K}}{\text{Jy sr}^{-1}} \\ &= \frac{\kappa}{\nu^2}I_\nu \quad \left[\frac{\text{mK}}{\text{Jy sr}^{-1}}\right]. \end{aligned} \tag{2}$$

The "telescope co-ordinates" have a one-to-one correspondence to cosmological co-ordinates, so that the measurement equation can be written

$$\begin{aligned} \tilde{V}(\mathbf{k}) = \frac{1}{\kappa}\int d^2\mathbf{r}_\perp dr_{||}\frac{dl}{dr_\perp}(l,\nu)\frac{dl}{dr_\perp}(m,\nu)\frac{d\nu}{dr_{||}}(\nu) \\ \times \nu^2(r_{||})T_b(\mathbf{r}_\perp,r_{||})A(\mathbf{r}_\perp,r_{||})\phi(r_{||})e^{-i\mathbf{r}\cdot\mathbf{k}}, \end{aligned} \tag{3}$$

where $\mathbf{k}$ is the Fourier-dual of $\mathbf{r}$[1], and we note that $\tilde{V}$ retains its original units of JyHz. It is common to approximate the conversion functions between angular/frequency and cosmological distance co-ordinates as linear transformations, so that their derivative is a constant. We note that this is only a good approximation when both the beam and bandwidth are "small enough" such that the true relations (given below) are approximately constant over the integration range. For now, we will make the first of these assumptions – that the beam is compact enough such that the interval dl corresponds to a roughly constant $dr_\perp$ over the integrated sky – and we will reserve the latter approximation for a later point.

Regardless, these conversion factors are often expressed as X and Y, and are given (at zenith) as follows:

$$X(\nu) \equiv \frac{dr_\perp}{dl} = D_M(\nu) \quad \left[\frac{\text{cMpc}}{\text{rad}}\right], \tag{4}$$

$$Y(\nu) \equiv \frac{dr_{||}}{d\nu} = \frac{c(1+z_\nu)}{H(z_\nu)\nu} \quad \left[\frac{\text{cMpc}}{\text{Hz}}\right], \tag{5}$$

where D_M is the transverse comoving distance (Hogg 1999), $H(z)$ is the Hubble parameter as a function of redshift and z_ν is the redshift corresponding to a frequency ν of 21 cm radiation, $z_\nu = \nu_{21}/\nu - 1$.

Thus, we may finally write the measurement equation in terms of cosmological co-ordinates as

$$\tilde{V}(\mathbf{k}) = \frac{1}{\kappa}\int d^3\mathbf{r}\ \nu^2\frac{T_b(\mathbf{r})A_\nu(\mathbf{r}_\perp)\phi_\nu}{X_\nu^2Y_\nu}e^{-i\mathbf{r}\cdot\mathbf{k}} \tag{6}$$

$$= \frac{1}{\kappa}\int\frac{d^3\mathbf{k}'}{(2\pi)^3}\ \widetilde{T_b}(\mathbf{k}')\widetilde{\Upsilon}(\mathbf{k}-\mathbf{k}'), \tag{7}$$

[1] i.e. the standard cosmological wave-number, incorporating an extra factor 2π as compared to the conversion of (u,v,η) to $\mathbf{r}$.

Assumption	Relevant Eqs.
Flat-sky approximation	All
Delay Approximation	All
Fourier-space beam, $\tilde{A}(\mathbf{k})$, much narrower than $P(\mathbf{k})$	17 –
$\frac{dx_\perp}{dl}(l,\nu) \approx \frac{dx_\perp}{dl}(0,\nu) \equiv X(\nu)$ for all ν in effective bandwidth, and all l in beam area.	6 –
Data composed only of 21 cm signal and thermal noise	11 –
Homogeneous, isotropic signal over the bandwidth	14 –
X, Y, ν^4 and $\Omega_{\rm pp}(\nu)$ are much broader in ν than ϕ^2	22, 38, 39, 43
Gaussian 21 cm field	30 –
$2\|\mathbf{k}\|$ outside window function regime	33 –

Table 1. Assumptions made in deriving normalisation and covariance.

with

$$\Upsilon(\mathbf{r}) = \nu^2 \frac{A_\nu(\mathbf{r}_\perp)\phi_\nu}{X_\nu^2 Y_\nu}. \tag{8}$$

Note that it is understood that ν as it appears in these equations is a direct conversion from the third component of $\mathbf{r}$, i.e. $r_{||}$. The second equation here is equivalent to the first, but expressed via the convolution theorem in Fourier-space. We use the usual cosmological convention, i.e.

$$\widetilde{T}(\mathbf{k}) = \int_{\mathbb{V}} d^3\mathbf{x} \;\; T(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}, \tag{9}$$

$$T(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \;\; T(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{10}$$

where $\mathbb{V}$ is the integration volume (in our case, its precise value is unimportant as it will be canceled in the final analysis).

2.2 Delay Spectrum

We wish to derive an *unbiased* estimator of the 21 cm power spectrum at a single mode $\mathbf{k}$, in the sense that the expectation value of the estimate should converge to the true cosmological value. This is not possible without specifying some statistical form for the various factors involved, in particular the brightness temperature field, T_b.

For the remaining calculations, we assume that the temperature field is given by a sum of signal and thermal noise. This neglects foregrounds, as well as other potential systematic errors (such as calibration errors) which may be both non-negligible in amplitude in comparison to the 21 cm signal and have non-zero mean. We justify this simplification on the grounds that we are interested in a foreground avoidance scheme, in which we only retain modes in which the expected signal dominates the foregrounds. Thus we explicitly have

$$\begin{aligned} T_b(\mathbf{r}) &= T_{21}(\mathbf{r}) + T_N(\mathbf{r}) \\ \widetilde{T}_b(\mathbf{k}) &= \widetilde{T}_{21}(\mathbf{k}) + \widetilde{T}_N(\mathbf{k}). \end{aligned} \tag{11}$$

where the second equality holds due to the linear nature of the components. In particular, two independent and redundant visibilities (i.e. measuring the same $\mathbf{k}$) will have the same value for $\widetilde{T}_{21}$, but different (and independent) values for $\widetilde{T}_N$.

An *unbiased* power spectrum estimate is then simple to obtain by using the cross-product of independent and redundant visibilities. To be clear, by *unbiased* we mean that the expectation of the estimate corresponds exactly to the cosmological power spectrum. To begin, we write the expectation of the cross-product of independent visibilities:

$$\left\langle \tilde{V}_i \tilde{V}_j^*(\mathbf{k}) \right\rangle = \frac{1}{\kappa^2} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \frac{d^3\mathbf{k}''}{(2\pi)^3} \left\langle \widetilde{T_b^i}(\mathbf{k}') \widetilde{T_b^{*j}}(\mathbf{k}'') \right\rangle \widetilde{\Upsilon}(\mathbf{k}-\mathbf{k}') \widetilde{\Upsilon}(\mathbf{k}-\mathbf{k}''). \tag{12}$$

Expanding the temperature field in terms of its components, we have

$$\begin{aligned} \left\langle \widetilde{T}'_{b,i} \widetilde{T}''^{*}_{b,j} \right\rangle &= \left\langle \widetilde{T}'_{21} \widetilde{T}''^{*}_{21} \right\rangle + \left\langle \widetilde{T}'_{21} \widetilde{T}''^{*}_{N,j} \right\rangle + \left\langle \widetilde{T}''^{*}_{21} \widetilde{T}_{N,i} \right\rangle + \left\langle \widetilde{T}''_{N,i} \widetilde{T}''^{*}_{N,j} \right\rangle \\ &= \left\langle \widetilde{T}'_{21} \widetilde{T}''^{*}_{21} \right\rangle, \end{aligned} \tag{13}$$

where the last three components are zero because the noise terms are independent of both signal and each other, and have mean zero. Now, the remaining term is precisely the diagonal power spectrum. To see this, we have

$$\begin{aligned} \langle \tilde{T_{21}} \tilde{T}'^{*}_{21} \rangle &= \int d^3\mathbf{x} d^3\mathbf{x}' \langle T_{21}(\mathbf{x}) T_{21}(\mathbf{x}') \rangle e^{-i\mathbf{k}\cdot\mathbf{x} + i\mathbf{k}'\mathbf{x}'} \\ &= \int d^3\mathbf{x} d^3\mathbf{r} \langle T_{21}(\mathbf{x}) T_{21}(\mathbf{x}+\mathbf{r}) \rangle e^{-i\mathbf{k}\cdot\mathbf{x} + i\mathbf{k}'(\mathbf{x}+\mathbf{r})} \\ &= \int d^3\mathbf{x} d^3\mathbf{r} \xi_{21}(\mathbf{r}) e^{-i\mathbf{k}\mathbf{x} + i\mathbf{k}'(\mathbf{x}+\mathbf{r})} \\ &= (2\pi)^3 \delta(\mathbf{k}-\mathbf{k}') \int d^3\mathbf{r} \xi_{21}(\mathbf{r}) e^{-i\mathbf{k}'\cdot\mathbf{r}} \\ &= (2\pi)^3 \delta(\mathbf{k}-\mathbf{k}') P_{21}(\mathbf{k}). \end{aligned} \tag{14}$$

Here, on the third line we have used the definition $\xi(r) \equiv \langle T(\mathbf{x}) T(\mathbf{x}+\mathbf{r}) \rangle$, which is often termed the "correlation" or "covariance" function, though we note that it is not strictly a covariance as here defined, unless $\langle T(\mathbf{x}) \rangle = 0$. We note also that the last equality follows from the equality

$$P(\mathbf{k}) \equiv \int d^3\mathbf{r} \;\; \xi(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}}, \tag{15}$$

which is only true if the field T is homogeneous (i.e. the expectation of its mean is constant with $\mathbf{x}$)[2]. In particular, we note that this is *not* strictly true of the 21 cm field, whose expected mean changes with frequency/redshift. This limits

[2] This condition is equivalently expressed by the relation

$$\xi(\mathbf{r}) = \langle T(\mathbf{x}) T(\mathbf{x}+\mathbf{r}) \rangle = \frac{1}{\mathbb{V}} \int d^3\mathbf{x} \; T(\mathbf{x}) T(\mathbf{x}+\mathbf{r}).$$

That is, by the identification of *ensemble average* with *spatial average*.

the bandwidth that is able to be employed while maintaining these derivations as appropriate.

Thus we have

$$\left\langle \tilde{V}_i \tilde{V}_j^*(\mathbf{k}) \right\rangle = \frac{1}{\kappa^2} \int d^3\mathbf{k}' P_{21}(\mathbf{k}') \widetilde{\Upsilon}^2(\mathbf{k}-\mathbf{k}'). \tag{16}$$

It is typically now argued that the square of the convolution kernel, Υ, is much more compact than the power spectrum, centered around zero. This allows us to evaluate the power spectrum at $\mathbf{k}$ and remove it from the integral:

$$\begin{aligned}\left\langle \tilde{V}_i \tilde{V}_j^*(\mathbf{k}) \right\rangle &= \frac{P_{21}(\mathbf{k})}{\kappa^2} \int d^3\mathbf{q} \widetilde{\Upsilon}^2(\mathbf{q}), \qquad (17)\\ &= \frac{P_{21}(\mathbf{k})}{\kappa^2} \int d^3\mathbf{r} \Upsilon^2(\mathbf{r}) \\ &= \frac{P_{21}(\mathbf{k})}{\Phi}. \qquad (18)\end{aligned}$$

with the second line following from Parseval's theorem, and

$$\Phi \equiv \frac{\kappa^2}{\int d^3\mathbf{r}\Upsilon^2(\mathbf{r})} \quad \left[\frac{\mathrm{mK}^2\mathrm{Mpc}^3}{\mathrm{Jy}^2\mathrm{Hz}^2}\right], \tag{19}$$

Thus we may construct an unbiased estimate of the cosmological power spectrum via

$$\widehat{P}_{21}(\mathbf{k}) = \Phi \tilde{V}_i \tilde{V}_j^*. \tag{20}$$

Note the units here; the visibilities retain their native units of JyHz, while the estimate of the power spectrum is in cosmological co-ordinates, $\mathrm{mK}^2\mathrm{Mpc}^{-3}$.

The normalisation, Φ, is a direct generalisation of that found in P17, and equivalent to that in Kolopanis et al. (2019), and may be written in full as

$$\Phi = \left(10^{-23}\frac{c^2}{2k_B}\right)^2 \left[\int d\nu \ \nu^4 \frac{\Omega_{\mathrm{pp}}(\nu)\phi^2(\nu)}{X_\nu^2 Y_\nu}\right]^{-1}, \tag{21}$$

with $\Omega_{\mathrm{pp}} = \int d^2\mathbf{l} \ A(\mathbf{l})$. Under the approximation that ϕ is significantly more peaked than all other functions of ν (and that it peaks at ν), this reduces to the familiar

$$\Phi \approx \left(10^{-23}\frac{c^2}{2k_B\nu^2}\right)^2 \frac{X_\nu^2 Y_\nu}{\Omega_{\mathrm{pp}}(\nu) B_{\mathrm{pp}}}, \tag{22}$$

3 COVARIANCE OF PS ESTIMATE

We turn now to estimating the uncertainty on an estimate of the delay power spectrum, as defined in the previous subsection. P14a and previous derivations have evaluated this variance by evaluating the power spectrum for visibilities that include only thermal noise. However, this overlooks a small but definite contribution to the total uncertainty which arises from the cross-correlation of signal power with the thermal noise power. We do not make this approximation in our derivation.

We derive the full covariance between $\mathbf{k}$ modes here, though previous derivations in the literature have quite reasonably focused on only the variance. We will also highlight the variance after the initial derivation.

The covariance of the estimated power spectrum is

$$\begin{aligned}\mathbf{C}_P &\equiv \mathrm{Cov}\left[\hat{\mathrm{P}}(\mathbf{k}), \hat{\mathrm{P}}(\mathbf{k}')\right] \\ &= \Phi^2 \mathrm{Cov}\left[\tilde{\mathrm{V}}_\mathrm{i}\tilde{\mathrm{V}}_\mathrm{j}^*, \tilde{\mathrm{V}}'_\mathrm{i}\tilde{\mathrm{V}}'^*_\mathrm{j}\right]. \qquad (23)\end{aligned}$$

Given our choice of temperature components introduced in Eq. 11 (i.e. a sum of a 21 cm and a thermal noise component) and the linearity of the Fourier operator, we have

$$\tilde{V}_i = \tilde{V}_{21} + \tilde{V}_{\mathrm{N},i}, \tag{24}$$

with $\tilde{V}_{\mathrm{N},i}$ the noise visibility constructed from the coherent average of n independent and redundant visibilities.

We then have

$$\begin{aligned}\mathbf{C}_P \equiv \quad & \Phi^2\mathrm{Cov}\left[\tilde{V}_i\tilde{V}_j^*, \tilde{V}'_i\tilde{V}'^*_j\right] \\ = \quad & \Phi^2\mathrm{Cov}\left[\left|\tilde{V}_{21}\right|^2 + \tilde{V}_{21}\tilde{V}^*_{\mathrm{N},j} + \tilde{V}^*_{21}\tilde{V}_{\mathrm{N},i} + \tilde{V}_{\mathrm{N},i}\tilde{V}^*_{\mathrm{N},j},\right. \\ & \left.\left|\tilde{V}'_{21}\right|^2 + \tilde{V}'^*_{21}\tilde{V}'_{\mathrm{N},j} + \tilde{V}'_{21}\tilde{V}'^*_{\mathrm{N},i} + \tilde{V}'^*_{\mathrm{N},i}\tilde{V}'_{\mathrm{N},j}\right]. \qquad (25)\end{aligned}$$

Taking the sum of the covariance of all pairs, we can dispense with any term which contains a single realisation of a noise term, as its expectation is zero. We recall that

$$\mathrm{Cov}\left[\mathrm{XY}, \mathrm{ZW}\right] = \langle \mathrm{XYZW}\rangle - \langle\mathrm{XY}\rangle\langle\mathrm{ZW}\rangle, \tag{26}$$

and write

$$\begin{aligned}\frac{\mathbf{C}_P}{\Phi^2} =& \mathrm{Cov}\left[|\tilde{\mathrm{V}}_{21}|^2, |\tilde{\mathrm{V}}'_{21}|^2\right] + 2\Re\left[\langle\tilde{\mathrm{V}}_{21}\tilde{\mathrm{V}}'^*_{21}\rangle\langle\tilde{\mathrm{V}}_\mathrm{N}\tilde{\mathrm{V}}'^*_\mathrm{N}\rangle\right] \\ &+ \langle\tilde{V}_\mathrm{N}\tilde{V}'^*_\mathrm{N}\rangle^2, \qquad (27)\end{aligned}$$

where $\Re$ denotes taking the real part of the argument.

The first term of this equation represents the intrinsic cosmic covariance of the 21 cm power spectrum, the last is the "noise power" covariance, and the middle term is a cross-term which we expect to be non-negligible when the 21 cm and noise power are equally matched. We now turn to derive explicit forms for each of the three terms.

3.1 Cosmic Covariance

Let's take the first term of Eq. 27:

$$\Phi^2\mathrm{Cov}\left[|\tilde{\mathrm{V}}_{21}|^2, |\tilde{\mathrm{V}}'_{21}|^2\right] = \Phi^2\langle|\tilde{\mathrm{V}}_{21}|^2|\tilde{\mathrm{V}}'_{21}|^2\rangle - \mathrm{P}_{21}\mathrm{P}'_{21}. \tag{28}$$

Here the last term follows directly from Eq. 17. Fully expanding the first term, we obtain

$$\begin{aligned}\langle|\tilde{V}_{21}|^2|\tilde{V}'_{21}|^2\rangle = \frac{1}{\kappa^4}\int & d^3\mathbf{x}_1 d^3\mathbf{x}_2 d^3\mathbf{x}_3 d^3\mathbf{r}_4 \Upsilon(\mathbf{x}_1)\Upsilon(\mathbf{x}_2)\Upsilon(\mathbf{x}_3)\Upsilon(\mathbf{x}_4) \\ & \times \langle T_1T_2T_3T_4\rangle e^{-i[\mathbf{k}_1\cdot(\mathbf{x}_1-\mathbf{x}_2)+\mathbf{k}'\cdot(\mathbf{x}_3-\mathbf{x}_4)]}. \qquad (29)\end{aligned}$$

To simplify this term, we will assume that the temperature field is Gaussian – an approximation that we have not made until this point. We note that the temperature field is definitely *not* Gaussian (Watkinson et al. 2018), nevertheless we expect that the Gaussian approximation should serve reasonably well in estimating the covariance of the power spectrum. We then have, via Wick's/Isserlis' theorem

$$\langle T_1T_2T_3T_4\rangle = \langle T_1T_2\rangle\langle T_3T_4\rangle + \langle T_1T_3\rangle\langle T_2T_4\rangle + \langle T_1T_4\rangle\langle T_2T_3\rangle. \tag{30}$$

Using repeated applications of the convolution theorem and the identity Eq. 15, along with the fact that $P(-k) = P(k)$, we find

$$\begin{aligned}\langle|\tilde{V}_{21}|^2|\tilde{V}'_{21}|^2\rangle =& \frac{P(\mathbf{k})P(\mathbf{k}')}{\kappa^4}\left[\frac{\kappa^4}{\Phi^2} + \left|\int\frac{d^3\mathbf{k}''}{(2\pi)^3}\widetilde{\Upsilon}(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')\widetilde{\Upsilon}(\mathbf{k}'')\right|^2\right. \\ & \left. + \left|\int\frac{d^3\mathbf{k}''}{(2\pi)^3}\widetilde{\Upsilon}(\mathbf{k}+\mathbf{k}'-\mathbf{k}'')\widetilde{\Upsilon}(\mathbf{k}'')\right|^2\right]. \qquad (31)\end{aligned}$$

When considering the total covariance, the first of these three terms cancels with the subtracted product of means. Furthermore, we note that only one of the integral terms will ever contribute for $|k|, |k'| > 0$, as one of them will have its "peaked" function $\widetilde{\Upsilon}$ much closer to its peak over the integration range. For the purposes of simplicity, we consider only correlations between $\mathbf{k}$ and $\mathbf{k'}$ in the same octant, under which only the second term remains (numerically, it is easy enough to include both terms, though for opposing octants, the results are perfectly symmetric). This leaves, for the full covariance:

$$\Phi^2 \mathrm{Cov}\left[|\tilde{V}_{21}|^2, |\tilde{V}'_{21}|^2\right] \approx \mathrm{P}(\mathbf{k})\mathrm{P}(\mathbf{k'}) \frac{\left|\int \frac{\mathrm{d}^3\mathbf{k''}}{(2\pi)^3} \widetilde{\Upsilon}(\mathbf{k} - \mathbf{k'} - \mathbf{k''})\widetilde{\Upsilon}(\mathbf{k''})\right|^2}{\left|\int \mathrm{d}^3\mathbf{x}\Upsilon^2(\mathbf{x})\right|^2} \tag{32}$$

The numerator here is sharply peaked at $\mathbf{k} = \mathbf{k'}$, which means there is very little correlation between distant k-modes.

When $\mathbf{k} = \mathbf{k'}$ we obtain a variance of

$$\Phi^2 \mathrm{Var}(|\tilde{V}_{21}|^2) \approx P_{21}^2(\mathbf{k}), \tag{33}$$

where the approximation is due to the initial assumption of a Gaussian temperature field, and is valid for values of $|\mathbf{k}|$ large enough that $2|\mathbf{k}|$ is outside the peak of the window function[3].

3.2 Noise Covariance

We now turn to the contribution to the total covariance which arises solely from the thermal noise (i.e. the last term of Eq. 27). We note that the noise statistics properly belong in visibility space – we take them to be independent per baseline and per frequency[4], and are assumed to be zero-mean complex Gaussian variables. While the noise fluctuations are natively in units of Jy, their rms variance, $V_{\mathrm{N,rms}}$ is typically expressed via a combination of equivalent brightness temperatures,

$$\begin{aligned} T_{\mathrm{sys}} &= T_{\mathrm{sky}} + T_{\mathrm{rcv}} \\ &= V_{\mathrm{N,rms}} \frac{\kappa}{\nu^2 \Omega_p(\nu)} \sqrt{\Delta\nu t_{\mathrm{int}} N_{\mathrm{pol}}}, \end{aligned} \tag{34}$$

with $\Delta\nu$ the bandwidth of a channel, t the integration time of the visibility and $\Omega_p(\nu)$ the integral of the beam over the sky. The Ω_p enters in order to make the correct unit conversion between Jy and mK, and is a *definition*: so long as the system temperature is measured by converting the inherent Jy units using this factor, then we must use this same factor to convert back to Jy. The factor $\Delta\nu t_{\mathrm{int}} N_{\mathrm{pol}}$ enters to maintain the naive association of the intrinsic sky and receiver temperatures to the "system temperature" which must account for the fact that the noise is reduced by the extra independent correlator samples from extra bandwidth, integration time and polarizations.

In the cosmological power spectrum case we were able to roughly ignore the fact that the frequency Fourier transform is in practice discrete, because the actual intensity value for each bin is considered to be the integral over the bin. The approximation that results from this lack of discretization is very much of secondary concern, and the discrete values can be read off from the differential continuous function. However, in the case of the noise, this is not possible – the value of the noise within a bin is not the integral of infinitesimally small sub-bins. Thus, we treat the fourier-transform explicitly as discrete:

$$\begin{aligned} \langle \tilde{V}_N(\mathbf{u},\eta)\tilde{V}_N^*(\mathbf{u},\eta')\rangle &= (\Delta\nu)^2 \sum_{j,k}^{N_f} \phi_j\phi_k \langle V_N(\mathbf{u},\nu_j)V_N^*(\mathbf{u},\nu'_k)\rangle e^{-2\pi i(\eta\nu_j - \eta'\nu_k)} \\ &= \frac{\Delta\nu}{\kappa^2 t_{\mathrm{int}} N_{\mathrm{pol}}} \sum_j^{N_f} \phi^2(\nu_j) T_{\mathrm{sys}}^2 \Omega_{\mathrm{p}}^2 \nu^4 e^{-2\pi i\nu_j(\eta-\eta')}. \end{aligned} \tag{35}$$

To simplify the expression, we are now free to approximate the discrete sum as an integral, and write

$$\langle \tilde{V}_N(\mathbf{u},\eta)\tilde{V}_N^*(\mathbf{u},\eta')\rangle = \frac{1}{\kappa^2 t_{\mathrm{int}} N_{\mathrm{pol}}} \int d\nu\ \phi_\nu^2 T_{\mathrm{sys}}^2(\nu)\nu^4 \Omega_{\mathrm{p}}^2 e^{-2\pi i\nu(\eta-\eta')}. \tag{36}$$

The covariance in detail is thus written

$$\begin{aligned} \mathbf{C}_P &= \left[\frac{\Phi}{\kappa^2 t_{\mathrm{int}} N_{\mathrm{pol}}} \int d\nu\ \phi_\nu^2 T_{\mathrm{sys}}^2(\nu)\nu^4\Omega_{\mathrm{p}}^2 e^{-2\pi i\nu(\eta-\eta')}\right]^2 \\ &= \left[\frac{1}{t_{\mathrm{int}} N_{\mathrm{pol}}} \frac{\int d\nu\ \phi_\nu^2 T_{\mathrm{sys}}^2(\nu)\nu^4\Omega_{\mathrm{p}}^2 e^{-2\pi i\nu(\eta-\eta')}}{\int d\nu\ \nu^4 X_\nu^{-2} Y_\nu^{-1} \Omega_{\mathrm{pp}}(\nu)\phi_\nu^2}\right]^2. \end{aligned} \tag{37}$$

If we are to make the approximation, as we did in Eq. 22, that ϕ is a much more peaked function of ν than any other factor in the integral, then we may extract them to yield

$$\mathbf{C}_P \approx \left[\frac{T_{\mathrm{sys}}^2 \Omega_{\mathrm{p}}^2 X_\nu^2 Y_\nu}{\Omega_{\mathrm{pp}}^2(\nu) B_{\mathrm{pp}}^2 t_{\mathrm{int}} N_{\mathrm{pol}}} \int d\nu\ \phi_\nu^2 e^{-2\pi i\nu(\eta-\eta')}\right]^2, \tag{38}$$

for which the variance is

$$\mathrm{Var}(P) \approx \left[\frac{T_{\mathrm{sys}}^2 \Omega_{\mathrm{p}}^2 X_\nu^2 Y_\nu}{\Omega_{\mathrm{pp}}(\nu) t_{\mathrm{int}} N_{\mathrm{pol}}}\right]^2. \tag{39}$$

Note that this expression differs from eg. P17, as it has no dependence on the bandwidth integral whatsoever. This can be thought of as saying that changing the taper function cannot change the signal-to-noise, as the same taper is applied to both signal and noise.

3.3 Signal-Noise Cross Covariance

For the cross-term, we require only to calculate

$$\begin{aligned} \langle \tilde{V}_{21}\tilde{V}'^*_{21}\rangle &= \frac{1}{\kappa^2}\int d^3\mathbf{x}_1 d^3\mathbf{x}_2 \Upsilon(\mathbf{x}_1)\Upsilon(\mathbf{x}_2)\langle T_1 T_2\rangle e^{-i[\mathbf{k}\cdot\mathbf{x}_1 - \mathbf{k'}\cdot\mathbf{x}_2)]} \\ &= \frac{1}{\kappa^2}\int d^3\mathbf{x}_1 d^3\mathbf{r}\Upsilon(\mathbf{x}_1)\Upsilon(\mathbf{x}_1+\mathbf{r})\xi(\mathbf{r}) e^{-i[\mathbf{x}_1\cdot(\mathbf{k}-\mathbf{k'}) - \mathbf{k'}\cdot\mathbf{r}]} \\ &= \frac{1}{\kappa^2}\int d^3\mathbf{x}_1 d^3\mathbf{k''}\Upsilon(\mathbf{x}_1)\widetilde{\Upsilon}(\mathbf{k'}-\mathbf{k''})P(\mathbf{k''}) e^{+i\mathbf{k''}\cdot\mathbf{x}_1} e^{-i\mathbf{x}_1\cdot\mathbf{k}} \\ &= \frac{1}{\kappa^2}\int d^3\mathbf{k''}\widetilde{\Upsilon}(\mathbf{k}-\mathbf{k''})\widetilde{\Upsilon}(\mathbf{k'}-\mathbf{k''})P(\mathbf{k''}). \end{aligned} \tag{40}$$

[3] For smaller values of $|\mathbf{k}|$, the variance rises to twice the quoted value.

[4] Note that this is an approximation; baselines that are physically close together, and frequencies that are close together may exhibit some correlations in practice.

Again, we find that the κ^2 factor cancels with the normalisation, Φ. Apart from this, this covariance expression is not easily reduced further.

However, for the variance, we find

$$\Phi\langle\tilde{V}_{21}\tilde{V}^*_{21}\rangle = P(\mathbf{k}). \tag{41}$$

3.4 Total Variance

Bringing together the results of the previous three subsections, we write down the total *variance* of the estimated power:

$$\mathrm{Var}(\hat{P}) = \left[P_{21}(\mathbf{k}) + \frac{1}{t_{\rm int}N_{\rm pol}}\frac{\int d\nu\ \phi_\nu^2 T_{\rm sys}^2(\nu)\nu^4\Omega_{\rm p}^2 e^{-2\pi i\nu(\eta-\eta')}}{\int d\nu\ \nu^4 X_\nu^{-2}Y_\nu^{-1}\Omega_{\rm pp}(\nu)\phi_\nu^2}\right]^2. \tag{42}$$

Under the assumption that ϕ is sharply peaked, this reduces to

$$\mathrm{Var}(\hat{P}) \approx \left[P_{21}(\mathbf{k}) + \frac{T_{\rm sys}^2\Omega_{\rm p}^2 X_\nu^2 Y_\nu}{\Omega_{\rm pp}(\nu)t_{\rm int}N_{\rm pol}}\right]^2. \tag{43}$$

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