1 Background

In calibration, one is trying to determine the true visibilities $V_{ij}^{\text{true}}$ at a given time and frequency given observed visibilities and antenna gains $g_i$:

$$V_{ij}^{\text{obs}} = g_i g_j^* V_{ij}^{\text{true}}.$$  \hfill (1)

Redundant baseline calibration schemes like Omnical use the fact that the same true visibility is observed by many baselines to calculate gains and model visibilities $V_{ij}$ that minimize the sum

$$\chi^2 = \sum_{\text{baselines}} |V_{ij}^{\text{obs}} - g_i g_j^* V_{ij}|^2.$$  \hfill (2)

However, the solution that minimizes that sum is not unique. That’s because, at least in the case of a single polarization, there are 4 extra degrees of freedom that can change the gains and visibilities while not affecting the product $g_i g_j^* V_{ij}$. Those are

- (1) Overall amplitude: $g_i \rightarrow A g_i$ and $V_{ij} \rightarrow V_{ij}/A^2$.
- (2) Overall phase: $g_i \rightarrow g_i e^{i\phi}$.
- (3) and (4) Phase slope (tip and tilt): for a co-planar array, if $\vec{\Phi} = (\Phi_X, \Phi_Y)$, $\vec{r}_i$ is the position of the $i$th antenna, and $\vec{d}_{ij} = \vec{r}_i - \vec{r}_j$, then $g_i \rightarrow g_i e^{i\vec{\Phi} \cdot \vec{r}_i}$ and $V_{ij} \rightarrow V_{ij} e^{-i\vec{\Phi} \cdot \vec{d}_{ij}}$. Here $\Phi_X$ and $\Phi_Y$ refer to cartesian directions, not polarizations. This is allowed because $d_{ij}$ is the same for all antenna pairs with the same separation (and thus visibility).

So what about in the case with two antenna polarizations and four visibility polarizations?

2 Calibrating $V^{xx}$ and $V^{yy}$ Separately

When $x$ and $y$ polarizations are calibrated separately, the system of equations for which we’re trying to minimize $\chi^2$ separates into two independent systems. It follows then that both $x$ and $y$ independently have the same 4 degeneracies, for a total of 8. This has been verified numerically by looking at the null-space of $\text{linca} \ A^T A$ matrix which has 8 corresponding zero eigenvalues.\footnote{See Liu et al. 2010, \url{https://arxiv.org/abs/1001.5268} and Zheng et al. 2014, \url{https://arxiv.org/abs/1405.5527}}
3 Calibrating $V^{xx}$, $V^{xy}$, $V^{yx}$, and $V^{yy}$ Together

In this case, we are trying to minimize a single $\chi^2$ for all gains and visibilities:

$$\chi^2 = \sum_{a,b,c,x,y} \sum_{\text{baselines}} \left| V_{ij}^{rad,obs} - g_i^a g_j^b V_{ij}^{rad} \right|^2. \quad (3)$$

When all four polarizations are calibrated together, the visibilities are all connected together through the gains. In this case, numerical experiments reveal that there are 6 degeneracies. The six independent free parameters that leave $\chi^2$ unchanged are:

- **(1) and (2)** Overall $x$ and $y$ amplitudes: $g_i^x \rightarrow A_x g_i^x$, $g_i^y \rightarrow A_y g_i^y$, $V_{ij}^{xx} \rightarrow V_{ij}^{xx}/A_x^2$, $V_{ij}^{xy} \rightarrow V_{ij}^{xy}/(A_x A_y)$, $V_{ij}^{yx} \rightarrow V_{ij}^{yx}/(A_x A_y)$, and $V_{ij}^{yy} \rightarrow V_{ij}^{yy}/A_y^2$. Equivalently, these can be recast as overall amplitude and relative amplitude of $x$ and $y$.

- **(3) and (4)** Overall $x$ and $y$ phases: $g_i^x \rightarrow g_i^x e^{i\phi_x}$, $g_i^y \rightarrow g_i^y e^{i\phi_y}$, $V_{ij}^{xx} \rightarrow V_{ij}^{xx}$, $V_{ij}^{xy} \rightarrow V_{ij}^{xy} e^{i(\phi_y - \phi_x)}$, $V_{ij}^{yx} \rightarrow V_{ij}^{yx} e^{i(\phi_x - \phi_y)}$, and $V_{ij}^{yy} \rightarrow V_{ij}^{yy}$. Equivalently, these can be recast as overall phase and relative phase of $x$ and $y$.

- **(5) and (6)** Phase slope (tip and tilt): same as in the single-polarization case (see §1).

While one might assume that that one can actually have two different phase slopes, $\Phi_x$ and $\Phi_y$, one for each polarization, this is actually not allowed. If, for example, $g_i^x \rightarrow e^{i\Phi_x} r_i^x$ and $g_i^y \rightarrow e^{-i\Phi_y} r_i^y$, then we’d have to transform the visibility $V_{ij}^{xy} \rightarrow V_{ij}^{xy} e^{-i(\Phi_x r_i^x - \Phi_y r_i^y)}$. This is not allowed in redundant calibration (and thus doesn’t preserve $\chi^2$) because the visibility is not longer just a function of displacement but also depends on the absolute position of the two antennas, unless $\Phi_x = \Phi_y$. As a result of this new understanding, we will have to reassess how we remove degeneracies in our redundant calibration code to include degeneracy removal in $x$ to $y$ relative phases and amplitudes.

4 4-Polarization Calibration Assuming $V_{ij}^{xy} = V_{ij}^{yx}$

Since pseudo-Stokes $V = -iV^{xy} + iV^{yx}$, the assumption that pseudo-Stokes $V$ is minimized corresponds to the assumption that $V_{ij}^{xy} = V_{ij}^{yx}$. If we include this assumption, then 5 of the 6 degeneracies are maintained:

- **(1) and (2)** Overall $x$ and $y$ amplitudes: $g_i^x \rightarrow A_x g_i^x$, $g_i^y \rightarrow A_y g_i^y$, $V_{ij}^{xx} \rightarrow V_{ij}^{xx}/A_x^2$, $V_{ij}^{xy} \rightarrow V_{ij}^{xy}/(A_x A_y)$, $V_{ij}^{yx} \rightarrow V_{ij}^{yx}/(A_x A_y)$, and $V_{ij}^{yy} \rightarrow V_{ij}^{yy}/A_y^2$. Equivalently, these can be recast as overall amplitude and relative amplitude of $x$ and $y$.

- **(3)** Overall phase: $g_i^x \rightarrow g_i^x e^{i\phi}$ and $g_i^y \rightarrow g_i^y e^{i\phi}$. If $\phi_x \neq \phi_y$ then we’d need $V_{ij}^{xy} e^{i(\phi_y - \phi_x)} = V_{ij}^{yx} e^{i(\phi_x - \phi_y)}$ which is only true when $\phi_x = \phi_y$ and $V_{ij}^{xy} = V_{ij}^{yx}$.

- **(4) and (5)** Phase slope (tip and tilt): same as in the single-polarization case (see §1).

The 5 degeneracies have been verified by numerical experiment. The effect of assuming that pseudo-Stokes $V$ is 0 is to remove the relative phase degeneracy between $x$ and $y$. If we make this assumption, we’d have to modify our degeneracy removal code to also correct for the relative scale of the $x$ and $y$ amplitudes.

---

2If one erroneously assumes that $V_{ij}^{xy} = V_{ij}^{yx}$, which we originally thought was equivalent to minimizing pseudo-Stokes $V$, one actually reduces the number of degeneracies to 4. These are not the same as the 4 in the single-polarization case. Instead, it can be shown that the first 4 degeneracies from the four-polarization case are preserved: overall $x$ and $y$ amplitudes and phases (or equivalently overall amplitudes and phases and relative amplitudes and phases $x$ to $y$). It’s the last two degeneracies, the tip-tilt, that are removed by this assumption. This is because if $V_{ij}^{xy} \rightarrow V_{ij}^{xy} e^{-i\Phi} d_{ij}$ and $V_{ij}^{yx} \rightarrow V_{ij}^{yx} e^{i\Phi} d_{ij}$, then the two cannot be equal unless $\Phi = 0$. This removal of the tip and tilt is unphysical since the array can actually have a tip and tilt—one reason not to make this assumption.