HERA Memo #30:
Redundant Calibration Degeneracies with Four Polarizations

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1 Background

In calibration, one is trying to determine the true visibilities $V_{ij}^{\rm true}$ at a given time and frequency given observed visibilities and antenna gains g_i:

$$V_{ij}^{\rm obs} = g_i g_j^* V_{ij}^{\rm true}. \tag{1}$$

Redundant baseline calibration schemes like Omnical use the fact that the same true visibility is observed by many baselines to calculate gains and model visibilities V_{ij} that minimize the sum

$$\chi^2 = \sum_{\rm baselines} \left| V_{ij}^{\rm obs} - g_i g_j^* V_{ij} \right|^2. \tag{2}$$

However, the solution that minimizes that sum is not unique. That's because, at least in the case of a single polarization, there are 4 extra degrees of freedom that can change the gains and visibilities while not affecting the product $g_i g_j^* V_{ij}$. Those are

- **(1)** Overall amplitude: $g_i \to A g_i$ and $V_{ij} \to V_{ij}/A^2$.
- **(2)** Overall phase: $g_i \to g_i e^{i\phi}$.
- **(3)** and **(4)** Phase slope (tip and tilt): for a co-planar array, if $\vec{\Phi} = (\Phi_X, \Phi_Y)$, $\vec{r}_i$ is the position of the ith antenna, and $\vec{d}_{ij} = \vec{r}_i - \vec{r}_j$, then $g_i \to g_i e^{i\vec{\Phi}\cdot\vec{r}_i}$ and $V_{ij} \to V_{ij} e^{-i\vec{\Phi}\cdot\vec{d}_{ij}}$. Here Φ_X and Φ_Y refer to cartesian directions, not polarizations. This is allowed because d_{ij} is the same for all antenna pairs with the same separation (and thus visibility).

So what about in the case with two antenna polarizations and four visibility polarizations?

2 Calibrating V^{xx} and V^{yy} Separately

When x and y polarizations are calibrated separately, the system of equations for which we're trying to minimize χ^2 separates into two independent systems. It follows then that both x and y independently have the same 4 degeneracies, for a total of 8. This has been verified numerically by looking at the null-space of `lincal` $\mathbf{A}^\top\mathbf{A}$ matrix which has 8 corresponding zero eigenvalues.[1]

[1]See Liu et al. 2010, `https://arxiv.org/abs/1001.5268` and Zheng et al. 2014, `https://arxiv.org/abs/1405.5527`

3 Calibrating V^{xx}, V^{xy}, V^{yx}, and V^{yy} Together

In this case, we are trying to minimize a single χ^2 for all gains and visibilities:

$$\chi^2 = \sum_{a,b\in x,y} \sum_{\text{baselines}} \left| V_{ij}^{ab,\text{obs}} - g_i^a g_j^{b*} V_{ij}^{ab} \right|^2 . \tag{3}$$

When all four polarizations are calibrated together, the visibilities are all connected together through the gains. In this case, numerical experiments reveal that there are 6 degeneracies. The six independent free parameters that leave χ^2 unchanged are:

- **(1)** and **(2)** Overall x and y amplitudes: $g_i^x \to A_x g_i^x$, $g_i^y \to A_y g_i^y$, $V_{ij}^{xx} \to V_{ij}^{xx}/A_x^2$, $V_{ij}^{xy} \to V_{ij}^{xy}/(A_x A_y)$, $V_{ij}^{yx} \to V_{ij}^{yx}/(A_x A_y)$, and $V_{ij}^{yy} \to V_{ij}^{yy}/A_y^2$. Equivalently, these can be recast as overall amplitude and relative amplitude of x and y
- **(3)** and **(4)** Overall x and y phases: $g_i^x \to g_i^x e^{i\phi_x}$, $g_i^y \to g_i^y e^{i\phi_y}$, $V_{ij}^{xx} \to V_{ij}^{xx}$, $V_{ij}^{xy} \to V_{ij}^{xy} e^{i(\phi_y - \phi_x)}$, $V_{ij}^{yx} \to V_{ij}^{yx} e^{i(\phi_x - \phi_y)}$, and $V_{ij}^{yy} \to V_{ij}^{yy}$. Equivalently, these can be recast as overall phase and relative phase of x and y
- **(5)** and **(6)** Phase slope (tip and tilt): same as in the single-polarization case (see § 1).

While one might assume that that one can actually have two different phase slopes, $\vec{\Phi}_x$ and $\vec{\Phi}_y$, one for each polarization, this is actually not allowed. If, for example, $g_i^x \to e^{i\vec{\Phi}_x \cdot \vec{r}_i}$ and $g_j^{y*} \to e^{-i\vec{\Phi}_y \cdot \vec{r}_j}$, then we'd have to transform the visibility $V_{ij}^{xy} \to V_{ij}^{xy} e^{-i(\vec{\Phi}_x \cdot \vec{r}_i - \vec{\Phi}_y \cdot \vec{r}_j)}$. This is not allowed in redundant calibration (and thus doesn't preserve χ^2) because the visibility is not longer just a function of displacement but also depends on the absolute position of the two antennas, unless $\vec{\Phi}_x = \vec{\Phi}_y$. As a result of this new understanding, we will have to reassess how we remove degeneracies in our redundant calibration code to include degeneracy removal in x to y relative phases and amplitudes.

4 4-Polarization Calibration Assuming $V_{ij}^{xy} = V_{ij}^{yx}$

Since pseudo-Stokes V $= -iV^{xy} + iV^{yx}$, the assumption that pseudo-Stokes V is minimized corresponds to the assumption that $V_{ij}^{xy} = V_{ij}^{yx}$.[2] If we include this assumption, then 5 of the 6 degeneracies are maintained:

- **(1)** and **(2)** Overall x and y amplitudes: $g_i^x \to A_x g_i^x$, $g_i^y \to A_y g_i^y$, $V_{ij}^{xx} \to V_{ij}^{xx}/A_x^2$, $V_{ij}^{xy} \to V_{ij}^{xy}/(A_x A_y)$, $V_{ij}^{yx} \to V_{ij}^{yx}/(A_x A_y)$, and $V_{ij}^{yy} \to V_{ij}^{yy}/A_y^2$. Equivalently, these can be recast as overall amplitude and relative amplitude of x and y
- **(3)** Overall phase: $g_i^x \to g_i^x e^{i\phi}$ and $g_i^y \to g_i^y e^{i\phi}$. If $\phi_x \neq \phi_y$ then we'd need $V_{ij}^{xy} e^{i(\phi_y - \phi_x)} = V_{ij}^{yx} e^{i(\phi_x - \phi_y)}$ which is only true when $\phi_x = \phi_y$ if $V_{ij}^{xy} = V_{ij}^{yx}$.
- **(4)** and **(5)** Phase slope (tip and tilt): same as in the single-polarization case (see § 1).

The 5 degeneracies have been verified by numerical experiment. The effect of assuming that pseudo-Stokes V is 0 is to remove the relative phase degeneracy between x and y. If we make this assumption, we'd have to modify our degeneracy removal code to also correct for the relative scale of the x and y amplitudes.

[2] If one erroneously assumes that $V_{ij}^{xy} = V_{ij}^{yx*}$, which we originally thought was equivalent to minimizing pseudo-Stokes V, one actually reduces the number of degeneracies to 4. These are *not* the same as the 4 in the single-polarization case. Instead, it can be shown that the first 4 degeneracies from the four-polarization case are preserved: overall x and y amplitudes and phases (or equivalently overall amplitudes and phases and relative amplitudes and phases x to y). It's the last two degeneracies, the tip-tilt, that are removed by this assumption. This is because if $V_{ij}^{xy} \to V_{ij}^{xy} e^{-i\vec{\Phi}\cdot\vec{d}_{ij}}$ and $V_{ij}^{yx*} \to V_{ij}^{yx*} e^{i\vec{\Phi}\cdot\vec{d}_{ij}}$, then the two cannot be equal unless $\vec{\Phi} = 0$. This removal of the tip and tilt is unphysical since the array can actually have a tip and tilt—one reason not to make this assumption.